Supertransient chaos in the two-dimensional complex Ginzburg-Landau equation

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We have shown that the two-dimensional complex Ginzburg-Landau equation exhibits supertransient chaos in a certain parameter range. Using numerical methods this behavior is found near the transition line separating frozen spiral solutions from turbulence. Supertransient chaos seems to be a common phenomenon in extended spatiotemporal systems. These supertransients are characterized by an average transient lifetime which depends exponentially on the size of the system and are due to an underlying nonattracting chaotic set. [S1063-651X(96)08806-X]

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I. INTRODUCTION

The dynamics of physical, chemical, and biological systems is often described by complicated, nonlinear partial differential equations (e.g., Navier-Stokes equations), and the general treatment of these equations, analytically and numerically, turns out to be only possible under special restrictions. A common approach to this systems uses the fact that near the threshold of an instability the nonlinearities are weak, and the modulations of a basic pattern can be described by an envelope function. For this function a differential equation can be derived (amplitude equation), which is easier to treat than the general equations, but with the disadvantage of being restricted to the vicinity of the threshold. The number of universal forms of these equations is limited by the classification of different linear instabilities [1]. This envelope formalism plays an important role in a variety of physical systems, such as oscillatory chemical reactions, Rayleigh-Bénard convection, or plasma waves. In this paper we consider the amplitude equation derived for an oscillatory uniform instability describing a system in the vicinity of a Hopf bifurcation in two spatial dimensions, which is commonly known as the 2D complex Ginzburg-Landau equation. For the complex amplitude function in the rescaled form it is

$$A_t = RA + (1+i\alpha)\Delta A - (1+i\beta)|A|^2 A, \tag{1}$$

where R, α, and β are real parameters. In the following we impose periodic boundary conditions

$$A(x,y,t) = A(x+L,y,t) = A(x,y+L,t)$$

and by means of the scaling transformation

$$(x,y) \to \left(\frac{2\pi}{L}x, \frac{2\pi}{L}y\right), \quad t \to \left(\frac{2\pi}{L}\right)^2 t,$$

$$A \to \frac{L}{2\pi}A, \quad R \to \left(\frac{L}{2\pi}\right)^2 R$$

we can restrict our investigation to the domain $\Omega = [0,2\pi]\times[0,2\pi]$. Since for the parameter R in Eq. (1) holds $R \sim L^2$ applying the rescaling transformation the parameter R represents a measure for the spatial extension of the medium and has a meaning comparable to the Reynolds number in the Navier-Stokes equations.

The 2D Ginzburg-Landau equation exhibits, analogously to its 1D version, a class of traveling plane waves

$$A(\mathbf{r},t) = a(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)], \tag{2}$$

where $|a(\mathbf{k})|^2 = R - \mathbf{k}^2$ and $\omega = (\alpha-\beta)\mathbf{k}^2 + R\beta$. The spatial homogenous solution (Stokes solution) with $\mathbf{k}=0$ becomes unstable at the Benjamin-Feir line $1+\alpha\beta=0$ (see Fig. 1); all other waves are losing their stability below [2].

Another type of solution is the spiral wave which has the form

$$A(r,\phi,t) = F(r)\exp\{i[-\omega t - m\phi + \psi(r)]\}; \tag{3}$$

written in polar coordinates (r,ϕ), $m = \pm 1$ is the topological charge. The functions F and ψ can be expressed in analytical form only for $r=0$ and $r\to\infty$, where $F(0)=\psi(0)=0$ and $\lim_{r\to\infty}F(r) = \sqrt{1-q^2}$, $q = \lim_{r\to\infty}\psi'(r)$ is the asymptotic wave number, which is a unique function of α and β [3]. Investigations about the stability of this solution can be found in [4].

Besides this regular behavior, the Ginzburg-Landau equation shows a variety of phenomena known from other dynamical systems such as spatiotemporal chaos [5], intermittency [6], and transient chaos [7,8]. We focus our interest in this paper on the latter.

Dynamical systems show deterministic chaotic motion not only as an asymptotic long-term behavior but also as a transient state before reaching a nonchaotic or chaotic attractor [9,10]. This transient chaos is caused by the existence of a nonattracting chaotic set (chaotic saddle) in phase space. Nearly every trajectory starting from random initial conditions wanders to this chaotic set and stays for some time in its vicinity, displaying chaotic motion. Then the trajectory leaves the saddle and settles down to an attractor, usually a periodic or quasiperiodic orbit.

The typical time τ (lifetime) of a trajectory in the vicinity of the chaotic saddle can be defined by the number $N(t)$ of

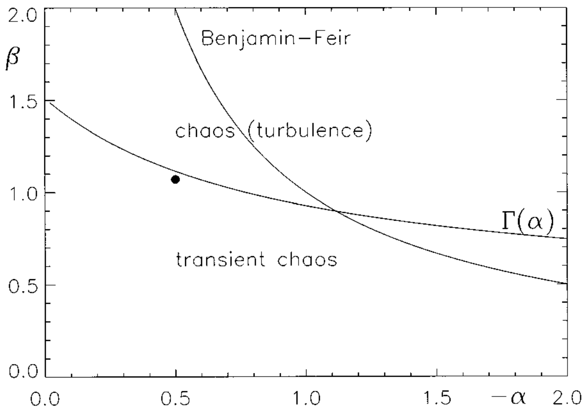

FIG. 1. Phase diagram of the 2D complex Ginzburg-Landau equation. The calculations presented here were carried out for $\alpha=-0.5$ and $\beta=1.07$ (filled circle).

trajectories, which still display chaotic motion at time t when starting at $t=0$ with N_0 different initial conditions

$$N(t)=N_0\ \exp[-t/\tau]. \tag{4}$$

For extended spatiotemporal systems it seems to be common that the lifetimes of the chaotic state can be extremely long. In these systems it was also found that the lifetime τ increases quickly with increasing system size, which will make it impossible to observe the nonchaotic attractor in a large system in practice, and such systems cannot be distinguished from systems containing a real chaotic attractor. If the lifetime depends exponentially on the system size the dynamics is also called supertransient chaos. This supertransient behavior is still not fully understood, and there are still only a few results available.

Investigations of the scaling of average transient lifetime τ in a 1D coupled map lattice (CML) were carried out in [11]. Depending on the coupling strength δ three different scaling behaviors were found. For weak coupling τ seems to be independent from the size L of the system. An increase of the coupling strength leads to a polynomial power law of the form $\tau\sim L^{\gamma}$ up to a critical value δ_c. Further increase of δ yields supertransient chaos,

$$\tau\sim\exp(aL^{\sigma})$$

with $\sigma\approx1$. Exponential scaling was also found in the investigation of the dynamics of complex interfaces [12], modeled by a 2D CML, with $\sigma=3/2$. A first example for supertransient behavior of a partial differential equation has been found by Wacker *et al.* in [13] for a special reaction-diffusion system. On the other hand, the system investigated in [14] shows a nearly linear growth of τ depending on its length.

The transient chaotic behavior in the Ginzburg-Landau equation was investigated modeled by a coupled map lattice in [7,8] in the parameter range $-2<\alpha<0$, $0<\beta<2$. Figure 1 shows the schematic phase diagram. The line $\Gamma(\alpha)$ is the transition line between transient and permanent chaos. Above $\Gamma(\alpha)$ there exists a chaotic attractor [in the region between $\Gamma(\alpha)$ and the Benjamin-Feir line BF this attractor coexists with periodic orbits], which changes into a chaotic saddle below $\Gamma(\alpha)$ as a result of a crisis [10]. In the transient region the trajectory finally settles down to a periodic attractor which is in most cases not a simple plane wave (2) but consists of a finite number of randomly distributed spirals. This final periodic state is often called a frozen state, because $|A|$ becomes time independent. Bohr *et al.* [7,8] investigated the scaling behavior of the transient lifetime in dependence of the parameter β for a fixed value of the parameter R. Approaching $\Gamma(\alpha)$ from below, they discovered an exponential dependence of the form $\tau\sim\exp\{[\Gamma(\alpha_0)-\beta]^{-2}\}$ for $\alpha_0=-1$. In contrast our aim is to examine the scaling behavior for fixed values of α and β to determine the dependence on the parameter R. As noted above, R can be interpreted as a length scale for the system.

II. NUMERICAL RESULTS

For the numerical simulation of the Ginzburg-Landau equation a pseudospectral method has been used applying a Fourier decomposition for the complex function $A(x,y,t)$ of the form

$$A(x,y,t)=\sum_{k_x,k_y\in\mathbb{Z}} a_{\mathbf{k}}(t)e^{i(k_xx+k_yy)},\quad \mathbf{k}=(k_x,k_y),$$

leading to a system of ordinary differential equations for the real and imaginary parts of the Fourier modes $a_{\mathbf{k}}(t)$.

All computations were performed on a CRAY Y-MP EL computer. Depending on the parameter R in the equation different resolutions consisting of 128×128, 64×64, or 32×32 gridpoints were chosen and the time integration was carried out by a fourth-order Runge-Kutta scheme.

We restrict ourselves to the parameter set $\alpha=-0.5$, $\beta=1.07$ (see filled circle in Fig. 1) and vary R within the interval $10\leqslant R\leqslant100$. In this range of R the final periodic attractor is for nearly all chosen initial conditions a single spiral [15].

In order to characterize the scaling of the transient lifetime with the system size it is necessary to perform calculations with several different initial conditions. As remarked

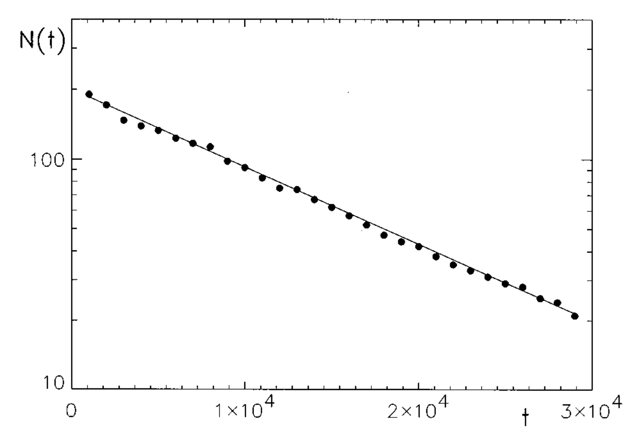

FIG. 2. Exponential decay of the number of trajectories displaying chaotic motion after time t. The graph was obtained by starting at $t=0$ with $N_0=200$ different initial conditions.

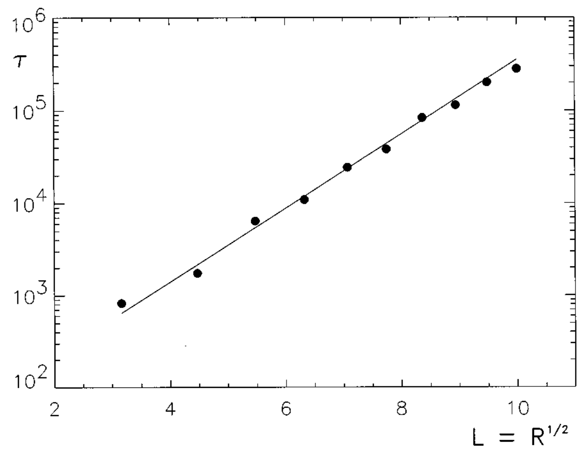

FIG. 3. Scaling of the average transient lifetime with the system size.

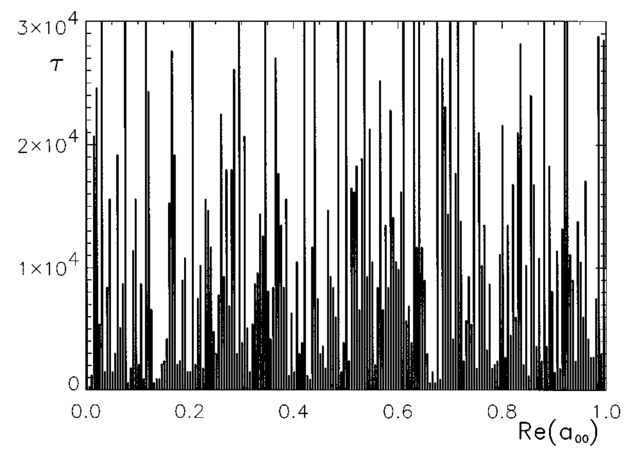

FIG. 4. Transient lifetimes on a 1D line segment in phase space.

above, the lifetime τ_μ of a transient chaotic trajectory generated by an initial condition $\{a_{\mathbf{k}}^{(\mu)}\}$ shows a strong dependence on this initial condition. To ensure that the initial conditions are in the vicinity of the chaotic saddle [16] we choose points on the chaotic attractor above $\Gamma(\alpha)$; the actual runs were carried out with this initial condition for β slightly below $\Gamma(\alpha)$. To let the computation stop automatically if the motion has become periodic we use that for a frozen state $\partial_t|A|=0$. It is easy to see that the same holds also for every individual mode of the Fourier decomposition, $\partial_t|a_{\mathbf{k}}(t)|=0$. We choose a time series of a nonzero mode $a_{\mathbf{k}_0}^t$ (here t is the discrete time $t=0,\Delta t,2\Delta t,\ldots$, with time step Δt) of a trajectory and a time interval T, which is short compared to the lifetime of the chaotic part of this trajectory, but large compared to the period of the final spiral solution. Then the condition $\nu_{n_\tau}=0$ for the quantity

$$\nu_n=\sum_{i=0}^{T/\Delta t-1}|r_{i,n}-r_{i+1,n}|, \quad r_{i,n}=|a_{\mathbf{k}_0}^{i\Delta t+nT}|$$

can be used to identify the time interval $I=[n_\tau T,(n_\tau+1)T]$, in which only periodic motion is displayed for the first time. As an approximation of the transient time we then have used $n_\tau T$.

Figure 2 shows the number $N(t)$ of trajectories which still display chaotic motion at time t [see Eq. (4)], obtained for $R=30$ and 200 different initial conditions. The exponential decay is clearly recognizable, the slope of the straight line gives the lifetime τ of the transient state.

For higher values of the parameter R (larger systems) the transient times turns out to be very large, and their determination using Eq. (4) fails because of the costly numerical calculations. For that reason we have estimated τ by the simple average $\tau=1/N\Sigma_\mu^N\tau_\mu$, using for every value of R ten different initial conditions.

The results are shown in Fig. 3. The average lifetime scales exponentially with the linear system size L

$$\tau\sim\exp(aL^\sigma)$$

with $a\approx1$, but due to the small number ($N=10$) of initial conditions it is impossible to specify the value of σ in order to compare it to other results ($\sigma=1$ in [11] or $\sigma=3/2$ in [12]). Nevertheless it can be stated that in the considered parameter range of R the 2D complex Ginzburg-Landau equation develops transient chaotic trajectories with lifetimes that scale exponentially with the size of the system, a behavior that this equation has in common with the spatially extended systems investigated in [11–13]. It should be mentioned that this exponential dependence on the system size may not hold for much greater values of R, when the final frozen state can consist of a larger number of spirals [8].

The geometric properties of the chaotic saddle responsible for supertransient chaos in a CML were investigated in [17]. It was found that in these systems the fractal dimension of the set of intersecting points of a one-dimensional line with the stable manifold of the chaotic saddle is close to 1. In order to get some information about the phase space structure of the Ginzburg-Landau system we use the sprinkle method described in [18], in which the stable manifold of the chaotic saddle is approximated by a set of initial conditions still displaying chaotic motion after a large time. Due to the high-dimensional phase space we restrict ourselves to a one-dimensional set of initial conditions distributed on a straight line. In practice we used again a point on the chaotic attractor slightly above $\Gamma(\alpha)$ and varied only the real part of the zero mode a_{00}^{Re} between 0 and 1 in steps of 0.005. In Fig. 4 the transient lifetimes τ are plotted versus the zero mode a_{00}^{Re}. Large values of τ indicate that the initial condition was close to the stable manifold. The intermingling appearance suggests a high-dimensional stable manifold of the chaotic saddle in analogy to the results obtained for the CML in [17], and which seems to be responsible for the occurrence of the supertransients in the Ginzburg-Landau equation.

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